

A compared analysis of the susceptibility in the $O(N)$ theory

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Abstract

The longitudinal susceptibility χ_L of the $O(N)$ theory in the broken phase is analyzed by means of three different approaches, namely the leading contribution of the $1/N$ expansion, the Functional Renormalization Group flow in the Local Potential approximation and the improved effective potential via the Callan-Symanzik equations, properly extended to $d = 4$ dimensions through the expansion in powers of $\epsilon = 4 - d$. The findings of the three approaches are compared and their agreement in the large N limit is shown. The numerical analysis of the Functional Renormalization Group flow equations at small N supports the vanishing of χ_L^{-1} in $d = 3$ and $d = 3.5$ but is not conclusive in $d = 4$, where we have to resort to the Callan-Symanzik approach. At finite N as well as in the limit $N \rightarrow \infty$, we find that χ_L^{-1} vanishes with J as $J^{\epsilon/2}$ for $\epsilon > 0$ and as $(\ln(J))^{-1}$ in $d = 4$.

I. INTRODUCTION

The $O(N)$ scalar theory is the simplest field theory with a continuous global symmetry, hence it has been widely used as a test to check new methods and analytic or numerical approximation schemes and its properties have been studied in detail. A very important property that is realized in this theory is the spontaneous symmetry breaking of the global symmetry, a phenomenon that plays a central role both in quantum field theory and in the study of phase transitions in statistical physics. In particular, the $O(N)$ theory is the simplest model suitable for studying the universal properties of the QCD chiral phase transition at finite temperature, as well as other aspects of the chirally broken phase of quark models.

The spontaneous breaking of a global symmetry is signaled by a nonzero value of the order parameter, which in the $O(N)$ theory corresponds to a nonvanishing vacuum expectation value of one component of the scalar field, and is typically associated with the appearance of massless excitations, the Goldstone bosons[1], whose propagation induce infrared divergences that can substantially modify the mean field indications. The nature and the strength of the infrared divergences is strictly related to the number of Euclidean dimensions d of the system and in particular the Coleman-Mermin-Wagner theorem, specifically formulated for spin models in[2]

and for quantum fields in [3], precludes the existence of a phase with conventional long range order, or spontaneous symmetry breaking in $d \leq 2$. When d increases, the infrared divergences become less severe and one has to resort to a suitable approximation scheme in order to deal with the infrared sector and extract finite answers for the physical observables.

A specific observable of the $O(N)$ theory where the effect of the Goldstone bosons is crucial, is the mass of the field that acquires a nonvanishing vacuum expectation value φ_0 , indicated as the longitudinal field. In fact, while it is possible to make use of general symmetry arguments to put constraints on the mass of the Goldstone bosons associated to the other $(N-1)$ transverse fields, the same is not true for the longitudinal field. When the scalar theory is coupled to a magnetic field J , the Ward-Takahashi identities, which relate the various correlation functions of the theory according to the symmetry properties of the problem, state that the inverse transverse susceptibility, defined as the inverse transverse propagator at zero momentum, is $\chi_T^{-1} = \Gamma_T(p=0) = J/\varphi_0$. Then, when the magnetic field J is turned off the Ward-Takahashi identities ensure the vanishing of $\Gamma_T(p=0)$, that is a vanishing Goldstone bosons mass [4].

There is no such constraint on the inverse longitudinal susceptibility $\chi_L^{-1} = \Gamma_L(p=0)$ and therefore on the mass of the longitudinal field. However, the development of Renormalization Group techniques provided a deeper understanding of the infrared structure of the theory, and for this specific problem in $2 < d < 4$, the relation $\chi_L^{-1} \propto J^{(4-d)/2}$ is predicted [5–7]. This scaling of the longitudinal mass has been repeatedly analyzed and confirmed in various approaches [8–12]. However the behavior of this quantity in $d = 4$ is different because in this case the power law scaling is replaced by much weaker logarithmic corrections. In [8] a few arguments are presented to show that χ_L^{-1} also vanishes in $d = 4$ and in [13], where the two point function of the $O(N)$ theory is studied in the framework of the Functional Renormalization Group, a numerical investigation suggests a vanishing mass in $d = 4$.

The Functional Renormalization Group is a reformulation of the Wilsonian Renormalization Group [14], based on the infinitesimal integration of momentum modes from a path integral representation of the theory with the help of a Wilsonian momentum cutoff. The resulting functional flow equations interpolate between the microscopic theory at short distances and the full quantum effective theory at large distances. Various realizations of the Functional Renormalization Group were developed and, among the most renowned, there are Polchinski's and Wetterich's formulations [15, 16] (for reviews see [17] [18]). Although this method is particularly suitable to analyze the critical domain and is especially powerful in the evaluation of the related critical exponents (see for instance [19–21] for an extended list of references), it also turns out to be an important tool in the study of nonperturbative features of the effective potential and effective action. For instance, the effective potential of the broken phase of scalar theories with its convex form (this convexity property, [22–24], does not show up in perturbation theory[25] and can only be recovered in some nonperturbative scheme [26]) has been largely investigated by means of Renormalization Group techniques [27–38],[13]. In particular, an improvement on the determination of the effective potential in the ordered phase

of the $O(N)$ theory is considered in [13], where a numerical computation of the longitudinal propagator which includes the wave function renormalization, i.e. momentum dependent corrections to the Local Potential approximation, is performed in $d = 4$ with the result that large wave function renormalization factors appear at least away from the critical region of the phase transition to the disordered phase.

In this paper we reconsider the study of the longitudinal mass in the broken phase of the $O(N)$ theory, in the framework of the Functional Renormalization Group, generalizing the case $d = 4$ examined in [13] to the range $2 < d \leq 4$, however not including momentum dependent corrections that are not crucial for our purpose and therefore limiting ourselves to the Local Potential approximation. We first analyze the problem in the simpler large N limit case and compare the results with those obtained at lowest order in the $1/N$ expansion. Then we extend our investigation to small values of N , also confronting the results with the Renormalization Group improved effective potential as obtained from the Callan-Symanzik equations. With this compared study, we gain analytic control on the problem which allows to overcome the limits of the numerical analysis, that are particularly evident in proximity of $d = 4$, and finally we get a more complete picture of this specific issue. In Sec. II we briefly go through the determination of the longitudinal susceptibility at the leading order in the $1/N$ expansion and in Sec. III we discuss the Functional Renormalization Group approach in the same limit. In Sec. IV the problem at small N is investigated by means of the Functional Renormalization Group techniques and of the perturbative Renormalization Group improvement. Conclusions are reported in Sec. V.

II. THE LARGE N LIMIT

We start by considering the $O(N)$ theory defined by the partition function

$$Z[J] = \int \mathcal{D}\Phi e^{-S[\Phi] + \int J \cdot \Phi} \quad (1)$$

where Φ is a N -component scalar field and the action $S[\Phi]$ reads

$$S[\Phi] = \int d^d x \left\{ \frac{1}{2} [\partial_\mu \Phi]^2 + \frac{M^2}{2} \Phi^2 + \frac{\lambda}{4!} [\Phi^2]^2 \right\}, \quad (2)$$

and we focus on the limit of large number of field components, $N \rightarrow \infty$. By using the standard technique of introducing an auxiliary field ρ in the partition function via the following constant multiplicative factor (where the integration contour in the complex ρ plane must be taken parallel to the imaginary axis)

$$\int \mathcal{D}\rho e^{\frac{3}{2\lambda} \int d^d x [\rho - (M^2 + \frac{\lambda}{6} \Phi^2)]^2} = \int \mathcal{D}\rho e^{\int d^d x \left[\frac{3}{2\lambda} \rho^2 - \frac{3M^2}{\lambda} \rho - \frac{1}{2} \rho \Phi^2 + \frac{M^2}{2} \Phi^2 + \frac{\lambda}{4!} (\Phi^2)^2 + \frac{3M^4}{2\lambda} \right]} \quad (3)$$

and then, by indicating with φ the longitudinal field component parallel to the source J and integrating over the remaining $(N - 1)$ components of Φ , the new effective action, functional of the fields φ and ρ , that appears in the partition function

$$Z[J] = \int \mathcal{D}\varphi \mathcal{D}\rho e^{-S_{eff}[\varphi, \rho] - \int d^d x (J, \varphi)} \quad (4)$$

is:

$$S_{eff}[\varphi, \rho] = \int d^d x \left[\frac{1}{2}(\partial_\mu \varphi)^2 + \frac{1}{2}\rho\varphi^2 + \frac{3M^2}{\lambda}\rho - \frac{3}{2\lambda}\rho^2 \right] + \frac{N-1}{2} \text{tr} \ln [(-\partial^2 + \rho) \delta^d(x-y)] \quad (5)$$

where we also included the coupling to the source J .

This procedure shows the explicit dependence on N and it is understood that, for large N , the square field φ^2 , the inverse coupling $1/\lambda$ and the effective action S_{eff} are proportional to N . The limit $N \rightarrow \infty$ is typically studied by keeping the product $(N\lambda)$ finite and performing the functional integral by means of the steepest descent method. In particular, by restricting to uniform configurations for φ and ρ , one gets the two coupled extremum equations:

$$\frac{\delta S_{eff}}{\delta \varphi} = \rho\varphi = J, \quad (6)$$

$$\frac{\delta S_{eff}}{\delta \rho} = -\frac{3}{\lambda}(\rho - M^2) + \frac{\varphi^2}{2} + \frac{N}{2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + \rho} = 0 \quad (7)$$

and the fluctuations around the solutions of Eqs.(6,7) are neglected because they are suppressed by inverse powers of N .

From Eqs.(6,7) at $J = 0$ one can immediately find the symmetric phase solution where $\varphi = 0$ and ρ is the selfconsistent square mass solution of the gap equation (7). Clearly this is possible only if $M^2 > M_c^2$, where

$$M_c^2 = -\frac{\lambda N}{6} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2}. \quad (8)$$

For $M^2 < M_c^2$ the system admits another solution and the broken phase is realized.

The broken phase, which will be considered below, corresponds to a nonvanishing field $\varphi \neq 0$ at $J = 0$. This, according to Eq. (6), implies $\rho = 0$ and at the same time Eq. (7) yields the value of the field φ_0 that corresponds to the minimum of the potential

$$\varphi_0^2 = -\frac{6M^2}{\lambda} + \frac{6M_c^2}{\lambda}. \quad (9)$$

Then we are able to rephrase Eqs.(6,7) at finite J , in terms of φ_0

$$\rho = \frac{J}{\varphi_0} \left(1 + \frac{\varphi - \varphi_0}{\varphi_0} \right)^{-1} \quad (10)$$

$$\varphi^2 - \varphi_0^2 = \frac{6}{\lambda} \rho - N \int \frac{d^d p}{(2\pi)^d} \left(\frac{1}{p^2 + \rho} - \frac{1}{p^2} \right) \quad (11)$$

and it is easy to check the dependence of the field φ on the source J , at least in $2 < d < 4$, where the integral in the right hand side of Eq. (11) is finite and proportional to $\rho^{\frac{d-2}{2}}$. After performing the integral, Eq. (11) becomes

$$\varphi^2 - \varphi_0^2 = \frac{6\rho}{\lambda} - N \frac{\Gamma(1 - d/2)}{(4\pi)^{d/2}} \rho |\rho|^{\frac{d-4}{2}}. \quad (12)$$

Eq. (12) can be explicitly solved for ρ , at least in the region $|\rho| \ll 1$, (which means close to the minimum of the potential, according to the saddle point equations (6,7) with $\varphi \neq 0$) where the term $\rho^{\frac{d-2}{2}}$ is dominant with respect to the linear term ρ ,

$$\rho = [\theta(\varphi^2 + \varphi_0^2) - \theta(\varphi_0^2 - \varphi^2)] \left(- (4\pi)^{d/2} \frac{|\varphi^2 - \varphi_0^2|}{N\Gamma(1 - d/2)} \right)^{\frac{2}{d-2}}. \quad (13)$$

The solution (13), when replaced in Eq. (6), $V'(\varphi) = \varphi\rho$, yields an analytic expression of the derivative of the potential in the region $|\rho| \ll 1$.

Incidentally when $M^2 = M_c^2$, i.e. $\varphi_0 = 0$, one finds:

$$\varphi \propto J^{\frac{d-2}{d+2}} \quad (14)$$

and the inverse of the exponent of J gives the correct critical index $\delta = (d+2)/(d-2)$. This picture is slightly modified in the broken phase with $\varphi_0 \neq 0$, again for $2 < d < 4$. In fact in this case it is convenient to define the shifted field $\varphi_{sh} = \varphi - \varphi_0$ so that in the left hand side of Eq. (11), for $J \rightarrow 0$ one can retain the $O(\varphi_{sh})$ term, neglecting $O(\varphi_{sh}^2)$ and, in the same way, the leading behavior of ρ is given by $\rho \sim J/\varphi_0$. Then, as before, in the right hand side of Eq. (11) one can retain only the leading term proportional to $\rho^{\frac{d-2}{2}}$ and finally one finds

$$\varphi_{sh} \propto J^{\frac{d-2}{2}}. \quad (15)$$

It is then straightforward to get the behavior of $(\partial\varphi(J)/\partial J)$ from the derivative of Eq. (15) for small J and $2 < d < 4$, by recalling that φ_0 in Eq. (9) is independent of J ,

$$\chi_L^{-1}(J) = \left(\frac{\partial\varphi}{\partial J} \right)^{-1} \propto J^{\frac{4-d}{2}}. \quad (16)$$

Then, clearly, for $2 < d < 4$ when the source J is turned off we have $\chi_L^{-1}(J=0) = 0$.

In $d = 4$, the integral in Eq. (11) contains a logarithmic divergence and the expression in Eq. (15) is modified by logarithmic corrections so that it is no longer straightforward to generalize Eq. (16) to $d = 4$ by following the same procedure. Therefore, we shall reconsider the problem by analyzing in detail the behavior of the integral, including the case $d = 4$. In particular we use Eqs.(10,11) to express $\rho = \rho(\varphi, J)$ and finally φ as a function of the source J : $\varphi = \varphi(J)$. Therefore, by deriving Eqs.(10,11) with respect to J and putting them together one has

$$\chi_L^{-1}(J) = \left(\frac{\partial \varphi(J)}{\partial J} \right)^{-1} = \rho + \frac{2\varphi^2}{\frac{6}{\lambda} + N I_2^{(d)}(\rho)} \quad (17)$$

where

$$I_2^{(d)}(\rho) = \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2 + \rho)^2} \quad (18)$$

The inverse longitudinal susceptibility is given by Eq. (17) in the limit $J \rightarrow 0$. Incidentally we note that the same result as in Eq. (17) is obtained as well, when computing the longitudinal mass $m_L^2(J) = \chi_L^{-1}(J)$ from the inverse propagator of the theory $\delta S_{eff}/(\delta \varphi \delta \varphi)$ (by treating ρ as $\rho(\varphi)$) at zero momentum. To extract the correct behavior of Eq. (17) when $J \rightarrow 0$ it is essential to analyze in detail the integral $I_2^{(d)}(\rho)$ including the case $d = 4$. We compute $I_2^{(d)}(\rho)$ by inserting an ultraviolet Euclidean cutoff Λ which protects the result from possible ultraviolet divergences:

$$\begin{aligned} I_2^{(d)}(\rho) &= C_d |\rho|^{\frac{d-4}{2}} \int_0^{\Lambda^2/|\rho|} dx \frac{x^{\frac{d-2}{2}}}{(x+1)^2} \\ &= C_d |\rho|^{\frac{d-4}{2}} \left[\frac{1}{1 + \frac{\Lambda^2}{|\rho|}} + \frac{(d-2)}{d} {}_2F_1 \left(1, \frac{d}{2}; \frac{d+2}{2}; -\frac{\Lambda^2}{|\rho|} \right) \right] \left(\frac{\Lambda^2}{|\rho|} \right)^{d/2} \\ &= C_d |\rho|^{\frac{d-4}{2}} \left[-\frac{d-2}{d} \Gamma(1-d/2) \Gamma(1+d/2) + \frac{2}{d-4} \left(\frac{|\rho|}{\Lambda^2} \right)^{\frac{4-d}{2}} \right. \\ &\quad \left. - \frac{4d}{(d-6)} \left(\frac{|\rho|}{\Lambda^2} \right)^{\frac{6-d}{2}} + \mathcal{O} \left[\left(\frac{|\rho|}{\Lambda^2} \right)^{\frac{8-d}{2}} \right] \right] \\ &= |\rho|^{\frac{d-4}{2}} \left(\frac{\Gamma(2-d/2)}{(4\pi)^{d/2}} + \frac{2}{(d-4)(4\pi)^{d/2} \Gamma(d/2)} \left(\frac{|\rho|}{\Lambda^2} \right)^{\frac{4-d}{2}} + \dots \right) \end{aligned} \quad (19)$$

where $C_d \equiv ((4\pi)^{d/2} \Gamma(d/2))^{-1}$, $\Gamma(x)$ is the Gamma function and ${}_2F_1(a, b; c; z)$ is the Gaussian Hypergeometric function.

We note that Eq. (19) when $d < 4$ does not present any ultraviolet divergences. The leading term is $[\Gamma(2-d/2)/(4\pi)^{d/2}] |\rho|^{\frac{d-4}{2}}$ and all the other terms in the development of Eq. (19) are suppressed by positive powers of $|\rho|/\Lambda^2$. If we take $\Lambda \rightarrow \infty$, only the leading term survives and we end up with the same result already shown in Eq. (16). However, due to the presence of the cutoff Λ , we are able to analyze the case $d = 4$. In fact one can easily check that, by

expanding the last line of Eq. (19) in powers of $\epsilon = 4 - d$, in the limit $d \rightarrow 4$ the divergent terms in ϵ cancel out leaving a logarithmic term :

$$I_2^{(4)}(\rho) = -\frac{1}{16\pi^2} \left(\ln \frac{|\rho|}{\Lambda^2} + 1 \right) \quad (20)$$

In this case the ultraviolet cutoff Λ appears logarithmically and we have to recall that for $d = 4$ the limit of infinite Λ corresponds to a trivial behavior of the theory that turns out to be noninteracting at any finite renormalization scale. Therefore we limit ourselves to the study of an effective theory with fixed ultraviolet cutoff Λ . Then the linear term in ρ in Eq. (17) is negligible and the inverse susceptibility goes to zero as $(|\ln(J)|)^{-1}$, when $J \rightarrow 0$. We shall analyze in more detail this aspect in Sec. IV.

III. FUNCTIONAL RENORMALIZATION GROUP AT LARGE N

In this Section we shall use the Functional Renormalization Group flow equations to analyze the theory in the large N limit. According to this approach one can follow the evolution of the running effective action Γ_k as a function of a momentum k scale which, starting from the ultraviolet cutoff Λ , goes down to the infrared limit $k = 0$. The flow of Γ_k starts in the ultraviolet region with an initial boundary condition given by the bare action and its final endpoint at $k = 0$ corresponds to the full effective action of the theory. The integration of the fluctuations down to the scale k , which generates the flow of Γ_k , is performed by modifying the action of the theory with an additional term, quadratic in the fields, that contains a k -dependent regulator R_k which acts like an infrared cutoff that effectively enables the integration of the modes above k . In order to fulfill the requirements that $\Gamma_{k=\Lambda}$ is the bare action and $\Gamma_{k=0}$ coincides with the full effective action, the regulator R_k must have the following asymptotic properties : $R_{k \rightarrow \infty} = \infty$ and $R_{k=0} = 0$. Then if $\Phi = (\Phi_1, \dots, \Phi_N)$, $\Gamma_k^{(2)}(q, -q; \Phi)_{a,b}$ is the second functional derivative:

$$\Gamma_k^{(2)}(q, -q; \Phi)_{a,b} = \frac{\delta^2 \Gamma_k[\Phi]}{\delta \Phi_a(q) \delta \Phi_b(-q)}, \quad (21)$$

$[R_k(q)]_{ab}$ is the k -dependent regulator and t is defined as $t = \ln(k/\Lambda)$, the flow equation reads:

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \text{Tr} \left(\partial_t R_k(q) \left[\Gamma_k^{(2)}[q, -q; \Phi] + R_k(q) \right]^{-1} \right) \quad (22)$$

where the trace is performed over the internal indices.

In order to analyze the behavior of the longitudinal susceptibility for the $O(N)$ theory, it is sufficient to approximate Eq. (22) to the so called Local Potential approximation, where we limit the form of $\Gamma_k[\Phi]$ by ignoring all possible scale dependent coefficients of the kinetic ($O(\partial^2)$) term and by fully neglecting $O(\partial^4)$ terms,

$$\Gamma_k[\Phi] = \int_x \left(U_k(|\Phi|) + \frac{1}{2} [\partial_\mu \Phi_a \partial_\mu \Phi_a] + O(\partial^4) \right). \quad (23)$$

In Eq. (23), due to the symmetry of the problem, the potential U_k depends on the modulus $|\Phi| = \sqrt{\Phi_a \Phi_a}$ and, according to our approximation, Eq. (22) reduces to a flow equation for the potential $U_k(|\Phi|)$, with ultraviolet boundary conditions fixed in Eq. (2) :

$$U_{k=\Lambda}(|\Phi|) = \frac{M^2}{2} \Phi^2 + \frac{\lambda}{4!} [\Phi^2]^2 \quad (24)$$

and the final infrared limit of the flow, $U_{k=0}(|\Phi|)$, corresponds to (an approximation of) the full effective potential.

The explicit form of the flow equation depends on the particular choice of the matrix R_k , and here it is convenient to take the following regulator that allows us to solve analytically the momentum integrals [39–42]:

$$[R_k(q)]_{ab} = (k^2 - q^2) \Theta(k^2 - q^2) \delta_{ab} \quad (25)$$

where $\Theta(x)$ indicates the Heaviside function. With the regulator in Eq. (25), the flow equation for $U_k(|\Phi|)$ is

$$\partial_t U_k(|\Phi|) = \frac{2C_d}{d} k^{2+d} \left[\frac{(N-1)}{k^2 + U'_k(|\Phi|)/|\Phi|} + \frac{1}{k^2 + U''_k(|\Phi|)} \right], \quad (26)$$

where C_d is defined after Eq. (19) and the primes indicate derivatives with respect to $|\Phi|$. The contribution of the $(N-1)$ Goldstone bosons and that of the longitudinal field in Eq. (26) are clearly evident.

The very simple structure of Eq. (26) allows us to analyze the large N limit, where the contribution of the longitudinal component is negligible and therefore, by deriving with respect to $|\Phi|$, we get

$$\partial_k U'_k(|\Phi|) = \frac{2C_d}{d} k^{1+d} \frac{N}{|\Phi|} \frac{U'_k(|\Phi|)/|\Phi| - U''_k(|\Phi|)}{(k^2 + U'_k(|\Phi|)/|\Phi|)^2}. \quad (27)$$

We recall that the inverse longitudinal susceptibility coincides with the square mass of the longitudinal field which is the second derivative of the effective potential evaluated at the vacuum expectation value of the field, i.e. at the minimum of the effective potential. Therefore in this framework it corresponds to $U''_{k=0}(|\Phi_0|)$ with $|\Phi_0|$ such that $U'_{k=0}(|\Phi_0|) = 0$. The flow equation in the form of Eq. (27) allows us to make some statements on $U''_{k=0}(|\Phi_0|)$.

Before doing that, we have first to recall the behavior of the running potential in the broken phase when $k \simeq 0$, which has repeatedly been described in various contexts [29],[31],[30],[32],[35–38],[13]. Essentially, it turns out that for k much smaller than the other characteristic scales of the problem, the derivative of the potential is very well approximated by $U'_k(|\Phi|) = -k^2 |\Phi| + O(k^3, |\Phi|^2)$ in the region $0 < |\Phi| \lesssim |\Phi_0|$, while for $|\Phi| \gtrsim |\Phi_0|$, $U'_k(|\Phi|)$ has already reached its infrared limit and is practically k -independent with the exception of a very narrow region around $|\Phi_0|$ where the two different branches merge. In this narrow region, no discontinuity is observed in $U'_k(|\Phi|)$, and in particular one has $U'_{k=0}(|\Phi_0|) = 0$. Accordingly,

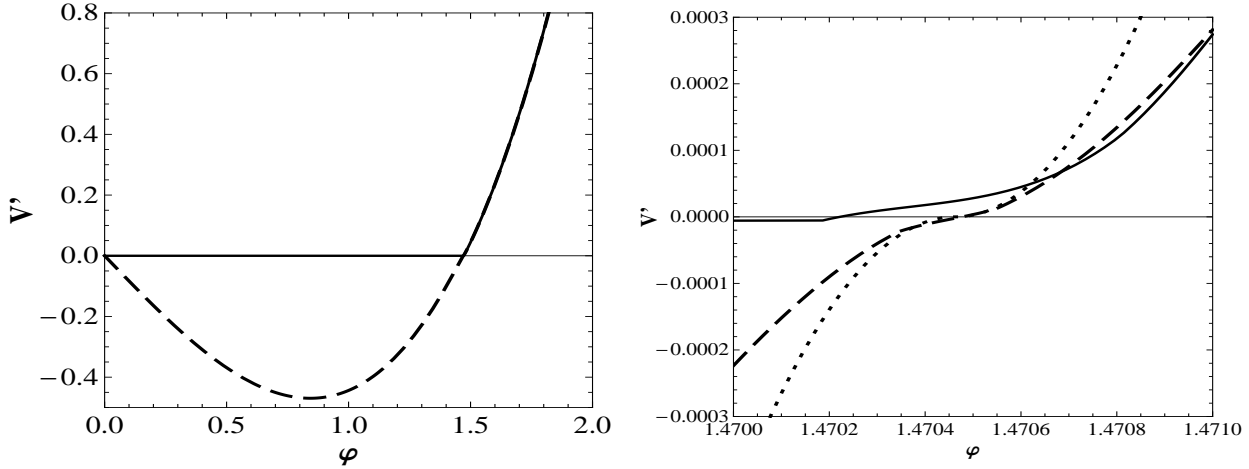


Figure 1: Left Frame. The derivative of the effective potential in the large N limit given in Eqs. (6) and (7) (dashed) compared with the Functional Renormalization Group $U'_k(|\Phi|/\sqrt{N})/\sqrt{N}$ at $k = 0.002$ (solid) for $d = 3$, $M^2 = -1$, $N\lambda = 2.4$, $\Lambda = 10$.

Right Frame. Zoom of the Left Frame in the region around $\varphi_0 = 1.47046$, with the inclusion of the analytic solution in Eq. (13) (dotted).

one finds that $U''_k(|\Phi|) \sim k^2$ for $0 < |\Phi| \lesssim |\Phi_0|$ and therefore, when $k \rightarrow 0$, its left limit (when $|\Phi| - |\Phi_0| \rightarrow 0^-$) vanishes. Our goal is to establish whether or not its right limit (when $|\Phi| - |\Phi_0| \rightarrow 0^+$) vanishes, i.e. whether or not $U''_{k=0}(|\Phi_0|)$ is discontinuous.

In order to clarify this point, we observe that according to the mentioned solution for $U'_k(|\Phi|)$ at $k \simeq 0$, the denominator in Eq. (27), computed at $|\Phi_0|$, vanishes as k^6 and, when combined with the other powers of k in Eq. (27), it gives the factor k^{d-5} which diverges in the infrared limit $k \rightarrow 0$ (in our problem $d \leq 4$). As the left hand side of Eq. (27) at $|\Phi_0|$ and for $k \rightarrow 0$ cannot be divergent, the numerator in the right hand side must vanish rapidly enough to compensate the diverging factor. Since the continuous function $U'_k(|\Phi_0|)$ vanishes as k^2 , it follows that also $U''_k(|\Phi_0|)$ must vanish when $k \rightarrow 0$ and therefore no discontinuity can appear because its presence would produce a divergence in the right hand side of the differential equation when $k \rightarrow 0$. This analysis indicates that in the large N limit the flow equations constrain the inverse susceptibility to vanish as $U''_k(|\Phi_0|) = -k^2 + O(k^3)$. Unfortunately this argument cannot be straightforwardly extended to the case of finite N , because of the presence of the longitudinal contribution in the differential flow equation.

On the other hand, one can try to extract the longitudinal susceptibility by numerically solving the Functional Renormalization Group flow. In fact the numerical resolution of the flow consists in the integration of a partial differential equation for the potential $U_k(|\Phi|)$ with fixed ultraviolet boundary conditions indicated in Eq. (24). The integration is carried out with the infrared scale that approaches $k = 0$ and the corresponding output of the flow provides a numerical approximation of the effective potential. Further details of the numerical integration

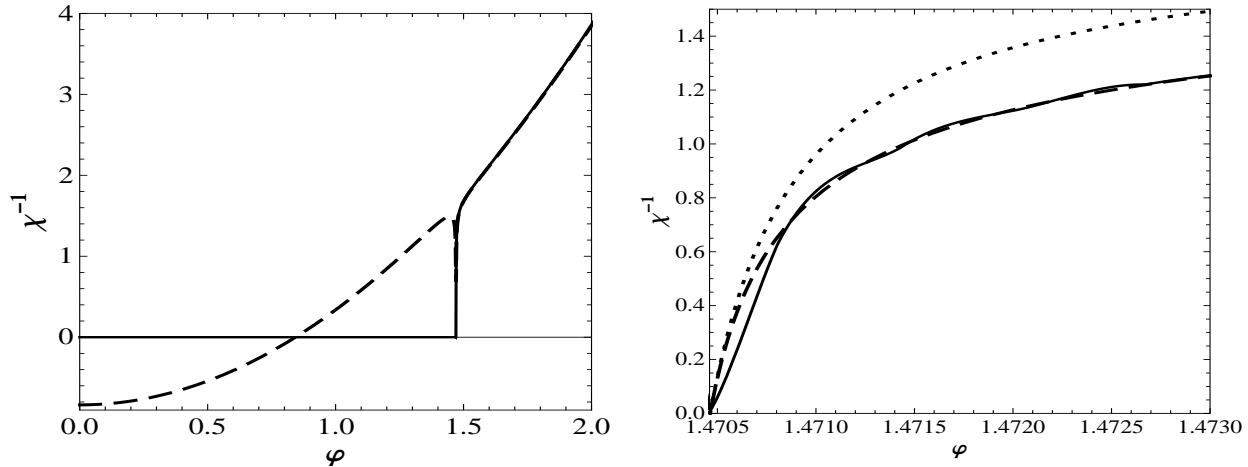


Figure 2: Left Frame. The inverse longitudinal susceptibility in the large N limit given in Eqs. (17) and (7) (dashed) compared with the Functional Renormalization Group $U_k''(|\Phi|/\sqrt{N})$ at $k = 0.002$ (solid) with the same parameters used in Fig. 1.

Right Frame. Zoom of the Left Frame in the region around $\varphi_0 = 1.47046$, with the inclusion of the analytic solution in Eq. (13) (dotted).

can be found in [13].

The numerical output of the Functional Renormalization Group flow can be compared with the solution of the saddle point equations obtained in the large N limit. In particular, we can compare U_k' and U_k'' with the first and second derivative of the potential obtained from Eqs. (6,17) where the parameter ρ is replaced according to Eq. (7). We can also check the analytic solution for ρ displayed in Eq. (13), which is valid only in a narrow region around the minimum. As a specific example, in Figs. 1 and 2 the first and second derivative of the potential are displayed for $d = 3$ and with the following set of bare parameters, $M^2 = -1$, $N\lambda = 2.4$. The ultraviolet cutoff is taken $\Lambda = 10$ for the Functional Renormalization Group and it is rescaled of a factor $2/3$ in Eqs. (6,7,17), because of the difference in the dependence of the renormalized parameters on Λ , as discussed in [13].

In the limit $k \rightarrow 0$, the Functional Renormalization Group flow produces a convex potential with a flat part in the region $|\Phi| \leq |\Phi_0|$, [27–38],[13], in agreement with a basic property of the effective potential [22–24]. Actually, the numerical routine adopted here fails to integrate the flow equations below a particular value of the infrared scale which corresponds to $k = 2 \cdot 10^{-3}$ in the example considered, and this explains the small difference from zero observed in the Functional Renormalization Group curve of Fig. 1 (right frame) for $|\Phi| \leq |\Phi_0|$. It is also well known that the convexity property of the potential is not recovered when using approximations based on an expansion around a saddle point, such as the loop expansion or the large N expansion considered in Sec. II. In these cases, it is necessary to consider the contribution of all the saddle points in order to obtain a convex potential [25]. This explains why in Fig. 1

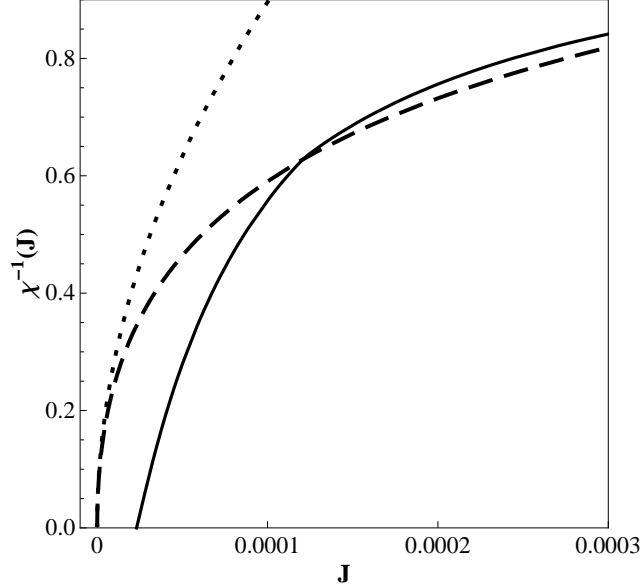


Figure 3: The inverse longitudinal susceptibility in the large N limit plotted as a function of the source J . All parameters are fixed as in Figs. 1 and 2. The same coding of Fig. 2 is used for dotted, dashed and solid curves.

we observe that at the leading order in the $1/N$ expansion, despite its claimed exactness, the effective potential does not present the expected convex shape in the region between $|\Phi| = 0$ and $|\Phi| = |\Phi_0|$.

On the other hand, for $|\Phi| \geq |\Phi_0|$ the computation performed by considering a unique saddle point is exact in the limit $N \rightarrow \infty$ and this explains the good agreement of the various curves above $|\Phi_0|$. We observe in the right frame of Figs. 1 and 2 that the analytic solutions for J (i.e. the derivative of the potential) and χ_L^{-1} (see Eqs. (6,7,17,13)), perfectly reproduce the numerical solution of the large N equations only for $|\Phi|$ very close to $|\Phi_0|$, as expected. From the plot of the curves in Fig. 2 it is also evident that the second derivative of the potential, i.e. the inverse susceptibility, vanishes at $|\Phi_0|$. Finally the plot of U_k'' computed via the Functional Renormalization Group is in good agreement with the other curves (compatibly with unavoidable numerical fluctuations) despite the flow is truncated before reaching $k = 0$.

Fig. 3 shows the inverse susceptibility as a function of the source J with $J \geq 0$ for the same set of parameters of Figs. 1 and 2. Again the numerical solution of the saddle point equations (6) and (7) is displayed together with the analytic solution in Eq. (13) and with the output of the flow equation. Fig. 3 shows the range of small values of J where the flow equation is no longer accurate. For larger values of J , the dashed and the solid lines coincide. In addition we notice that the analytic solution obtained by replacing ρ in Eq. (17) with the result of

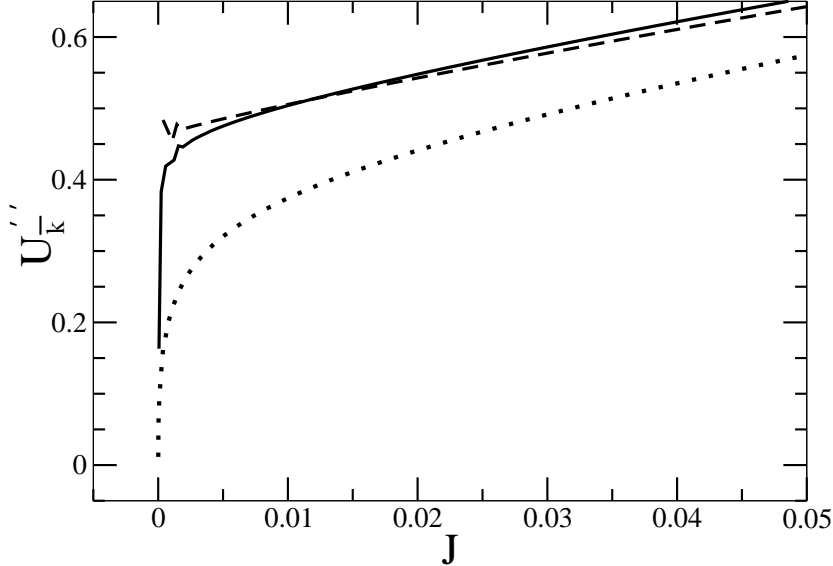


Figure 4: U_k'' plotted versus J for $d = 3$ and $\bar{k} = 5 \cdot 10^{-4}$ (dotted) , $d = 3.5$ and $\bar{k} = 10^{-2}$ (solid), $d = 4$ and $\bar{k} = 2 \cdot 10^{-2}$ (dashed). Values of the other parameters : $M^2 = -1$, $\lambda = 2.4$ and $\Lambda = 10$.

Eq. (13), and which displays the scaling behavior indicated in Eq. (16), does in fact coincide with the full numerical solution only for extremely small values of J . In this very small range, the Functional Renormalization Group flow equation solution has already deviated from the correct behavior and a more efficient algorithm becomes necessary to push k closer to zero to obtain a better agreement in this region and to recover the scaling behavior of Eq. (16).

IV. SUSCEPTIBILITY AT FINITE N

We start the analysis of the problem at finite N by looking at the full flow equation of the potential in Eq. (26) with both contributions of the transverse and longitudinal fields that introduce the dependence of the flow on two different scales, namely the running masses of these two fields. As already noticed it is not easy to put analytic constraints on the longitudinal mass when $k \rightarrow 0$ due to the presence of these two separate contributions.

We turn to the numerical resolution of the flow equation to extract some physical indications in the infrared limit. In particular, we are interested in the behavior of the inverse susceptibility, i.e. the mass of the longitudinal field. Let us first consider the case of $N = 4$ and bare parameters $M^2 = -1$, $\lambda = 2.4$ and cutoff $\Lambda = 10$. In $d = 3$, the flow is stable down to $\bar{k} = 5 \cdot 10^{-4}$ and the corresponding solution U_k'' is displayed in Fig. 4 (dotted line) as a function of the source $J = U_k'$, with $J \geq 0$. The other two curves in Fig. 4 respectively

correspond to $d = 3.5$ (solid), where the lowest reached infrared value is $\bar{k} = 10^{-2}$, and $d = 4$ (dashed), with $\bar{k} = 2 \cdot 10^{-2}$ and the bare parameters have the same values as in the $d = 3$ case.

Due to the finite value of \bar{k} , the three curves in Fig. 4 do not reach the origin ($J = 0, U'' = 0$). However, while for smaller d the curves clearly approach the origin, the same is not true when d is at the upper critical dimension. In particular, when $d = 3$ (dotted) the infrared cutoff k can be pushed very close to $k = 0$ and the corresponding curve in Fig. 4 smoothly bends toward the origin. When $d = 3.5$ (solid) we already observe numerical fluctuations in the plot but even in this case there is a change of slope and the curve approaches the origin. In $d = 4$ although the infrared cutoff is not much larger than for $d = 3.5$, the numerical fluctuations are stronger and make the deep infrared region ($k \rightarrow 0$) inaccessible. The corresponding curve (dashed) does not show any bending toward the origin and it is not possible to establish from the plot whether or not the second derivative of the potential vanishes at $J = 0$ and $k = 0$. In fact, as we shall see below by applying standard Renormalization Group techniques, such problem is related to the fact that power law corrections are replaced by smoother logarithmic corrections in $d = 4$, which turn out to be more difficult to treat numerically. For better understanding this point, we now proceed to the study of the theory at finite N and for d close to 4, in the framework of the Callan-Symanzik approach.

To this end, it is convenient to rewrite the potential as

$$V(\phi) = \frac{\tau}{12}\phi^2 + \frac{\phi^4}{24\lambda}, \quad (28)$$

where we have introduced the rescaled field ϕ

$$\phi^2 \equiv \lambda\Phi^2, \quad (29)$$

and the "temperature" τ

$$\tau = \frac{6}{\lambda}(M^2 - M_c^2), \quad (30)$$

is normalized in such a way that it vanishes at the critical value $M^2 = M_c^2$ that separates the symmetric and the broken phase. It is understood that M_c^2 includes corrections with respect to Eq. (8) which is obtained in the $N \rightarrow \infty$ limit. We are specifically interested in the broken phase and therefore we take $\tau < 0$.

We also recall that the coupling λ has dimension Λ^{4-d} , expressed in units of the ultraviolet cutoff Λ , and the corresponding dimensionless coupling is defined as $u = \lambda\Lambda^{d-4}$. By defining $t = \ln \frac{\mu}{\Lambda}$ as the logarithm of the scale μ , (we avoid to use the same notation of the running momentum k adopted for the Functional Renormalization Group flow equations due to the different roles of the two scales), the β -functions of the various parameters in the potential, as given by perturbation theory at lowest order, are ($\epsilon = 4 - d$) :

$$\beta = \frac{du}{dt} = -\epsilon u + C_d \frac{N+8}{3} u^2, \quad (31)$$

$$\frac{d(M^2 - M_c^2)}{dt} = C_d \frac{N+2}{3} u(M^2 - M_c^2). \quad (32)$$

Note that Φ at lowest order has no correction and therefore no t dependence. A word of caution concerning the usual way of obtaining these β -functions has to be said: the fact that the $O(N)$ model in the broken phase contains more than one scale requires a more careful treatment, [43], and we shall come back again to this point. From Eqs. (29,30,31,32) we get

$$\frac{d\tau}{dt} = -2C_d\tau \quad (33)$$

$$\gamma_\phi = \frac{1}{\phi} \frac{d\phi}{dt} = \frac{\epsilon}{2} + \frac{1}{2u} \frac{du}{dt} = \frac{C_d}{2} \frac{N+8}{3} u. \quad (34)$$

By solving the differential equations related to these β -functions, one can determine the evolution of the various parameters with the scale μ , after having set the boundaries at the bare values when $\mu = \Lambda$. Below, the dependence of the various variables on the parameter t is explicitly displayed while it is omitted for the bare variables, which correspond to the value $t = 0$. So the dimensionless coupling is

$$u(t) = ue^{-\epsilon t} \left[1 - \frac{C_d u(N+8)}{3\epsilon} (1 - e^{-\epsilon t}) \right]^{-1} \quad (35)$$

while the dimensionful coupling $\lambda(t) = u(t) \mu^\epsilon$ reads

$$\frac{1}{\lambda(t)} = \frac{1}{\lambda} + \frac{C_d(N+8)}{3\epsilon} (\mu^{-\epsilon} - \Lambda^{-\epsilon}) \quad (36)$$

and the temperature τ is

$$\tau(t) = \tau \left(\frac{u}{u(t)} e^{-\epsilon t} \right)^{\frac{6}{N+8}}. \quad (37)$$

Finally, the renormalization factor $\xi(t)$ of the rescaled field ϕ is

$$\xi(t) = e^{\int_0^t dt' \gamma_\phi(t')} = \left(\frac{u}{u(t)} e^{-\epsilon t} \right)^{-1/2} \quad (38)$$

We are now able to evaluate the Renormalization Group improved potential $V_{RGI}(\phi)$, solution of the Callan-Symanzik equation. This is achieved by replacing in Eq. (28) the bare quantities with the corresponding scale dependent parameters:

$$V_{RGI}(\phi) = \frac{\tau(t)}{12} \xi(t)^2 \phi^2 + \frac{1}{24\lambda(t)} \xi^4(t) \phi^4. \quad (39)$$

The derivative of $V_{RGI}(\phi)$ (in this Section the prime indicates derivative with respect to ϕ) is:

$$V'_{RGI}(\phi) = \frac{1}{6} \xi^2(t) \phi \left[\tau(t) + \frac{1}{\lambda(t)} \xi(t)^2 \phi^2 \right] = \frac{J}{\sqrt{\lambda}}. \quad (40)$$

We can also determine the second derivative of $V_{RGI}(t)$ which, after recalling the definition of J in Eq. (40), reads

$$V_{RGI}'' = \frac{J}{\phi\sqrt{\lambda}} + \frac{1}{3\lambda(t)}\xi(t)^4\phi^2. \quad (41)$$

Let us now go back to a somewhat delicate point. The model under investigation has two characteristic scales $M_1(\phi)$ and $M_2(\phi)$, respectively associated with the mass of the longitudinal field and with the Goldstone bosons mass: $M_1^2(\phi) = \lambda V''(\phi)$ and $M_2^2(\phi) = \lambda V'(\phi)/\phi$. If these two scales are comparable, then the usual procedure to compute $V_{RGI}(\phi)$, that we have outlined above, is essentially correct. On the other hand, when these scales are significantly decoupled, new aspects, that we have overlooked up to now, appear [43]. It is important to observe that some perturbative diagrams contributing to the longitudinal two point function are affected by infrared divergences due to the loops of Goldstone bosons and a reliable estimate of $V_{RGI}''(\phi)$ is obtained only when the Goldstone contributions are included in the evaluation of the Renormalization Group functions. Indeed, the Goldstone bosons have zero mass in absence of external source and, when J is turned on, their square mass is proportional to J , while the longitudinal field starts with a nonvanishing bare mass at $t = 0$ and, at least in the $N \rightarrow \infty$ case, its renormalized square mass is proportional to $J^{\epsilon/2}$ when J approaches zero. Therefore, in the evaluation of the β -function we expect that the deep infrared region is dominated by the smaller scale $M_2(\phi)$, so that the contribution of the larger scale $M_1(\phi)$ can be neglected. Accordingly, we select $\mu^2 = M_2^2(\phi) = J\sqrt{\lambda}/\phi$ with $t = \ln(\mu/\Lambda)$ and Eqs. (31,32) are replaced by

$$\frac{du}{dt} = -\epsilon u + C_d \frac{N-1}{3} u^2, \quad (42)$$

$$\frac{d(M^2 - M_c^2)}{dt} = C_d \frac{N-1}{3} u(M^2 - M_c^2), \quad (43)$$

Consequently Eq. (33) becomes $d\tau/dt = 0$ so that its solution in (37) now becomes $\tau(t) = \tau$, while the factor $(N+8)$ has to be replaced by $(N-1)$ in Eqs. (34,35,36) and Eq. (38) is formally unchanged.

Since our analysis concerns the infrared region corresponding to large negative t , at this point it is essential to recall the different behavior of the theory in $2 < d < 4$ and $d = 4$, due to the presence, in the former case, of the Wilson-Fisher fixed point which merges with the Gaussian fixed point when $d = 4$. Therefore we have to treat the two cases separately and we start by considering $2 < d < 4$. Then, from Eq. (42) one finds the Wilson-Fisher fixed point

$$u^* = \frac{3\epsilon}{C_d(N-1)} \quad (44)$$

(that vanishes when $\epsilon \rightarrow 0$) and the scaling of $u(t)$ and $\xi(t)$ near this fixed point is

$$\frac{1}{u(t)} \simeq \frac{1}{u^*} + \left(\frac{1}{u} - \frac{1}{u^*} \right) e^{\epsilon t} \quad (45)$$

$$\xi(t) \simeq \left(\frac{u}{u^*}\right)^{-1/2} e^{\frac{\epsilon}{2}t}. \quad (46)$$

From the infrared behavior of the coupling in Eq. (45), one can write for the corresponding dimensionful quantity, after neglecting subleading terms:

$$\frac{\lambda}{\lambda(t)} \simeq \frac{u}{u^*} e^{-\epsilon t}. \quad (47)$$

Now we can focus on the inverse susceptibility which, due to the rescaling of ϕ in Eq. (29), is $\chi_L^{-1}(J) = \sqrt{\lambda}(\partial J/\partial\phi) = \lambda V''_{RGI}$ and it can be read from Eq. (41) after inserting the explicit form of the running parameters. Then, according to Eqs. (46, 47), for small values of the source the linear term in J in the right hand side of Eq. (41) can be neglected with respect to the other one, and one finds:

$$\chi_L^{-1}(J) = \lambda V''_{RGI} \simeq \frac{u^*}{3u} e^{\epsilon t} \phi^2 = \frac{u^*}{3u} \left(\frac{J\sqrt{\lambda}}{\phi\Lambda^2}\right)^{(4-d)/2} \phi^2 \quad (48)$$

In the limit $J \rightarrow 0$ in Eq. (48), the field ϕ can be replaced with ϕ_0 (up to subleading terms), so the inverse susceptibility vanishes as $\chi_L^{-1}(J) \propto J^{\epsilon/2}$ when $J \rightarrow 0$, which is the same behavior found in Eq. (16) in the limit of large N .

The comparison with the large N case can be pushed forward by looking at Eq. (40) and by including the scale dependence displayed in Eqs. (44,45,46,47). For large N we get

$$\frac{J}{\sqrt{\lambda}} = \frac{\phi}{6} \left(\tau + \frac{\phi^2}{\lambda}\right) \left[\left(\frac{J\sqrt{\lambda}}{\phi\Lambda^2}\right)^{(4-d)/2} \frac{u^*}{u}\right] = \frac{\phi}{6\lambda} \left(\tau + \frac{\phi^2}{\lambda}\right) \left[\frac{3\epsilon}{C_d N} \left(\frac{J\sqrt{\lambda}}{\phi}\right)^{(4-d)/2}\right]. \quad (49)$$

It is not difficult to see that Eq. (49) coincides with the $N \rightarrow \infty$ result in Eq. (12), at least in the region of small J and $0 < \epsilon < 1$, where one can neglect the linear term in ρ and approximate $\Gamma(1 - d/2) = \Gamma(-1 + \epsilon/2) \simeq -2/\epsilon$. Incidentally, we notice that the particular choice of the Goldstone mass as the infrared scale and of the β -function in (42), has selected only the Goldstone bosons contribution, which is also dominant in the large N approximation. In this sense, it is not surprising to find the same result in Eqs. (12) and (49).

Finally we analyze the problem in $d = 4$, i.e. $\epsilon = 0$, where the coupling is dimensionless ($\lambda = u$) and, as is well known and as observed in the $N \rightarrow \infty$ case in Sec II, the divergences associated to the ϵ^{-1} poles are typically replaced by logarithms of the Renormalization Group scale. In addition, the main difference with respect to the problem in $d < 4$ is that the Wilson-Fisher and the Gaussian fixed points coincide in $d = 4$, $u^* = 0$, and this in turn implies that, when the ultraviolet cutoff is removed, one ends up with a noninteracting theory. In practice we can observe this 'triviality' property directly from the coupling constant, as obtained by integrating Eq. (42) in the limit $\epsilon \rightarrow 0$ (below we define $h = \frac{u}{16\pi^2} \frac{N-1}{3}$)

$$u(t) = \frac{u}{1 - h t} \sim \frac{u}{h |t|}, \quad (50)$$

where the last term is obtained in the region of large negative t . In fact, from the second term in Eq. (50) (i.e. before considering its approximation for large negative t that is displayed in the right hand side of Eq. (50)), it is evident that the limit of infinite ultraviolet cutoff $\Lambda \rightarrow \infty$, with fixed bare coupling u , corresponds to $t \rightarrow -\infty$ for any finite infrared scale μ and therefore one finds $u(t = -\infty) = 0$, i.e. the coupling vanishes at any finite μ . At the same time, we notice that the noninteracting limit $u(t = -\infty) = 0$ is recovered also at finite Λ , but for a vanishing infrared scale $\mu = 0$ and, in conclusion, a nonvanishing coupling $u(t) \neq 0$ requires finite values of t , i.e. finite infrared and ultraviolet scales.

Then, going back to our problem, from Eqs. (42,43,38), in $d = 4$ and for large negative but finite t , we find $\tau(t) = \tau$ and $\xi(t) \sim (h|t|)^{-1/2}$. With these results we can immediately specialize Eq. (40) to $d = 4$:

$$\frac{J}{\sqrt{u}} = \frac{\phi}{6} \left[\tau (h|t|)^{-1} + \frac{\phi^2}{u} (h|t|)^{-1} \right], \quad (51)$$

and, in the same way, the second derivative of the potential in Eq. (41) becomes

$$\chi_L^{-1}(J) = u V_{RGI}'' = \frac{\sqrt{u}J}{\phi} + \frac{\phi^2}{3h|t|}, \quad (52)$$

where J has been inserted according to Eq. (51). Eq. (52) shows that the inverse longitudinal susceptibility, as obtained from V_{RGI} , depends only on the ratio of the Renormalization Group scales Λ and μ through the logarithm t . Then, since the infrared scale is taken equal to the Goldstone boson mass, $\mu^2 = M_2^2(\phi) = J\sqrt{\lambda}/\phi$, we recover the logarithmic dependence of $\chi_L^{-1}(J)$ on the source J (after neglecting the first term in the right hand side of Eq. (52) which vanishes linearly with J). It is also straightforward to realize that in the large N limit, Eq. (52) reproduces Eq.(17) with the loop integral displayed in Eq. (20).

Therefore, our conclusion in $d = 4$ is that $\chi_L^{-1}(J)$ vanishes as $(\ln(J))^{-1}$ when the external source is turned off, exactly as it happens in the large N limit. However it must be remarked that, because of the logarithmic dependence of our observable on the ratio of two scales, the limit $J \rightarrow 0$ cannot be disentangled from the limit $\Lambda \rightarrow \infty$ and this means that the results obtained in the limit $J \rightarrow 0$ are those of a noninteracting theory.

Before finishing we would like to investigate the possibility of observing a nonvanishing longitudinal susceptibility in $d = 4$. According to the above analysis, it is clear that formally the only way of achieving this result is to stay with a finite value of t , by keeping a finite Λ and also a finite infrared scale which for the moment could be taken as a generic value of the field: $\mu^2 = \phi^2/u$. If μ is taken much larger than M_1 and M_2 introduced above, then the problem related to the presence of two different infrared scales is no longer present and we can use the β -functions in Eqs. (31) and (32). This leads to a rescaling of the factor $h \rightarrow h_r = h(N+8)/(N-1)$ in Eq. (50) and to the following explicit form of Eqs. (37) and (38):

$$\tau(t) \sim \tau(h_r|t|)^{\frac{6}{N+8}} \quad ; \quad \xi(t) \sim (h_r|t|)^{-1/2}. \quad (53)$$

As a consequence Eq. (40) now becomes

$$\frac{J}{\sqrt{u}} = \frac{\phi}{6} \left[\tau (h_r |t|)^{-\frac{N+2}{N+8}} + \frac{\phi^2}{u} (h_r |t|)^{-1} \right]. \quad (54)$$

The minimum of V_{RGI} is located at $\phi_0 \neq 0$, such that $J(\phi_0) = 0$ in Eq. (54) :

$$\phi_0^2 = -\tau u \left(\frac{h_r}{2} \left| \ln \left(\frac{\phi_0^2}{u \Lambda^2} \right) \right| \right)^{\frac{6}{N+8}}. \quad (55)$$

The square magnetization ϕ_0^2/u displayed in Eq. (55), or equivalently the temperature τ , corresponds to a finite quantity that can be used to set the infrared scale in this calculation. Then Eq. (41), computed at $\phi = \phi_0$ (i.e. at $J = 0$) and with ϕ_0 replaced by τ according to Eq. (55), gives (up to logarithmic corrections), [4],

$$\chi_L^{-1} \simeq -\frac{\tau u}{3} \left(\frac{h}{2} \left| \ln \left(\frac{-\tau}{\Lambda^2} \right) \right| \right)^{-\frac{N+2}{N+8}}. \quad (56)$$

Eq. (56) provides a nonvanishing χ_L^{-1} , but it is evident that this result can only be obtained at the price of constraining t to a finite nonvanishing value, which in turn means that we are dealing with an interacting theory, effectively defined for values of the momentum smaller than Λ , and where the effects of the Goldstone bosons are screened by the finite infrared scale.

V. CONCLUSIONS

We studied the longitudinal susceptibility in the broken phase of the $O(N)$ theory, both in the simpler case of the large N limit and in the general case of finite N . In this analysis we compared the results of three different approaches, namely the leading contribution of the $1/N$ expansion, the Functional Renormalization Group flow equations in the Local Potential approximation and the improved effective potential via the Callan-Symanzik equations, properly extended to four dimensions through the expansion in powers of ϵ .

In the large N limit, the Functional Renormalization Group correctly reproduces a convex potential with a flat part in the region $|\Phi| \leq |\Phi_0|$, while this does not occur in the potential obtained in the large N expansion. In fact, the latter potential is computed by performing an asymptotic expansion of the path integral in $Z[J]$ around a single saddle point configuration. In order to recover the convexity of the effective potential for $|\Phi| \leq |\Phi_0|$, the contribution of the other saddle points has to be considered. The virtue of the Functional Renormalization Group is that it takes into account by construction all of these contributions. On the other hand, for $|\Phi| \geq |\Phi_0|$, the agreement of the two approaches is excellent.

According to the saddle point analysis performed in the large N limit, the inverse longitudinal susceptibility χ_L^{-1} vanishes with the external source J as $\chi_L^{-1} \propto J^{\epsilon/2}$, for $2 < d < 4$.

We have also shown that this scaling law is limited to a very narrow region around $J = 0$. In $d = 4$ the power law is replaced by the logarithm of the ratio of the ultraviolet cutoff and the infrared scale associated to the source J and $\chi_L^{-1}(J)$ vanishes with J as $(\ln(J))^{-1}$.

At finite N , the Functional Renormalization Group numerical analysis shows the vanishing of χ_L^{-1} for $d = 3$ and $d = 3.5$. In $d = 4$ the region around the minimum of the potential is affected by strong numerical fluctuations and, as a consequence, it is not possible to conclude from the numerical analysis whether or not χ_L^{-1} has a discontinuity at $|\phi_0|$. This point is clarified by means of the Callan-Symanzik approach. Within this latter framework we analyzed the longitudinal susceptibility for $\epsilon \geq 0$ and recovered the same behavior already found in the limit $N \rightarrow \infty$, namely χ_L^{-1} vanishing with J as $J^{\epsilon/2}$ for $\epsilon > 0$ and as $(\ln(J))^{-1}$ in $d = 4$, in agreement with the results of [8].

More generally, we find a full correspondence of our Renormalization Group improved expressions of the derivatives of the effective potential in the Callan-Symanzik approach when $N \rightarrow \infty$, with the same quantities computed at the leading order in the large N expansion. Finally, in this approach it is clearly illustrated how the behavior of the inverse longitudinal susceptibility in $d = 4$ is simultaneously related to the vanishing of the Goldstone boson mass and to the limit $\Lambda \rightarrow \infty$ which leads to a noninteracting scalar theory.

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